# A MINIMAL LAMINATION OF THE UNIT BALL WITH SINGULARITIES ALONG A LINE SEGMENT

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ABSTRACT. We construct a sequence of compact embedded minimal disks in the unit ball in Euclidean 3-space whose boundaries are in the boundary of the ball and where the curvatures blow up at every point of a line segment of the vertical axis, extending from the origin. We further study the transversal structure of the minimal limit lamination and find removable singularities along the line segment and a non-removable singularity at the origin. This extends a result of Colding and Minicozzi where they constructed a sequence with curvatures blowing up only at the center of the ball, Dean's construction of a sequence with curvatures blowing up at a prescribed discrete set of points, and the classical case of the sequence of re-scaled helicoids with curvatures blowing up along the entire vertical axis.

## 0. Introduction

In the study and classification of Minimal Surfaces an important question is what are the possible singular sets for limits of sequences of embedded minimal surfaces. The global problem in $\mathbf{R}^3$ is understood by results of Colding and Minicozzi [CM2]-[CM5], where all singularities are removable at least when the sequence is simply connected, and by the work of Meeks and Rosenberg in [MR] where they explain why the singular set is perpendicular to the limit foliation.

In contrast, for the local case in $\mathbf{R}^3$, Colding and Minicozzi in [CM1] prove the existence of a sequence of embedded minimal disks with boundaries in a sphere and with curvatures blowing up only at the center of the ball, where there is a non-removable singularity. A result of Dean, [BD], extends this example by constructing a sequence with curvatures blowing up at a prescribed discrete set of points. There is also the well known case of the sequence of re-scaled helicoids with curvature blowing up along the entire $x_3$-axis. Meeks and Weber [MW] construct singular sets that are properly embedded $C^{1,1}$-curves. Hoffman and White [HW], and Calle and Lee [CL] also give a variational way to compute examples.

In this paper we construct a sequence of compact embedded minimal disks in the unit ball in $\mathbf{R}^3$ whose boundaries are in the boundary of the ball and where the curvatures blow up at every point of a line segment of the negative $x_3$-axis. This sequence converges to a minimal limit lamination. We study the transversal structure of the limit lamination and find a foliation by parallel planes in the lower hemisphere and a leaf in the upper hemisphere that spirals into the $\{x_3 = 0\}$ plane, such that there are removable singularities at every point along the line segment of the negative $x_3$-axis but the singularity at the origin cannot be removed.

We will follow the structure of Colding and Minicozzi's result in [CM1]. The key difference in our approach here is that we alter the domain and the Weierstrass data used in [CM1] to create not just one singularity converging to the origin, but a sequence of singularities that converges to a line segment extending from the origin. In addition, the construction of the limit lamination uses a convergence result from [CM5]. And a Bernstein-type theorem is used to obtain the foliation by parallel planes in the lower hemisphere of the unit ball.

Our main result is Theorem 2 below, which constructs our sequence of compact embedded minimal disks in the unit ball with boundaries in the boundary of the ball and describes the limit lamination. Theorem 1 first constructs a sequence of compact embedded minimal disks with the necessary curvature and boundary properties.

**Theorem 1.** *There exists a sequence of compact embedded minimal disks* $0 \in M_N \subset \mathbf{R}^2 \times [-1/2, 1/2] \subset \mathbf{R}^3$, *each containing the vertical segment* $\{(0,0,t) \,|\, |t| \leq 1/2\} \subset M_N$, *with the following properties:*

(a) $\forall p \in \{(0,0,t) \,|\, -1/2 \leq t \leq 0\}, \lim\limits_{N \to \infty} |A_{M_N}|^2(p) = \infty$.

(b) $M_N \setminus \{(0,0,t)| \,|t| \leq 1/2\} = M_{1,N} \cup M_{2,N}$ *for multi-valued graphs* $M_{1,N}$, $M_{2,N}$ *over the* $\{x_3 = 0\} \setminus \{0\}$ *punctured plane.*

(c) $\sup_N \sup_{M_N \setminus B_\delta} |A_{M_N}|^2 = C_\delta < \infty$ *for all* $\delta > 0$ *and some constant* $C_\delta$ *depending on* $\delta$, *and where* $B_\delta$ *is a* $\delta$-*neighborhood of* $\{(0,0,t) \,|\, -1/2 \leq t \leq 0\}$ .

(d) *The boundary* $\partial M_N$ *lies outside a fixed cylinder* $\{(x_1, x_2, x_3)| \, x_1^2 + x_2^2 \leq r_0^2, -1/2 < x_3 < 1/2\}$ *where* $r_0$ *does not depend on* $N$. *Also, in each horizontal slice* $\{x_3 = t\} \cap M_N$, *for* $-1/2 \leq t \leq 0$ *(i.e. below the* $\{x_3 = 0\}$ *plane) the distance from the* $x_3$-*axis to* $\partial M_N$ *goes to infinity as* $N \to \infty$.

Figure 1 shows a diagram of horizontal slices of $M_{1,N}$, $M_{2,N}$.

The boundary properties of the sequence in Theorem 1(d) allow us to intersect with a smaller ball that is contained in the cylinder, pass to a subsequence, and then scale to obtain the sequence in Theorem 2 below that has the same
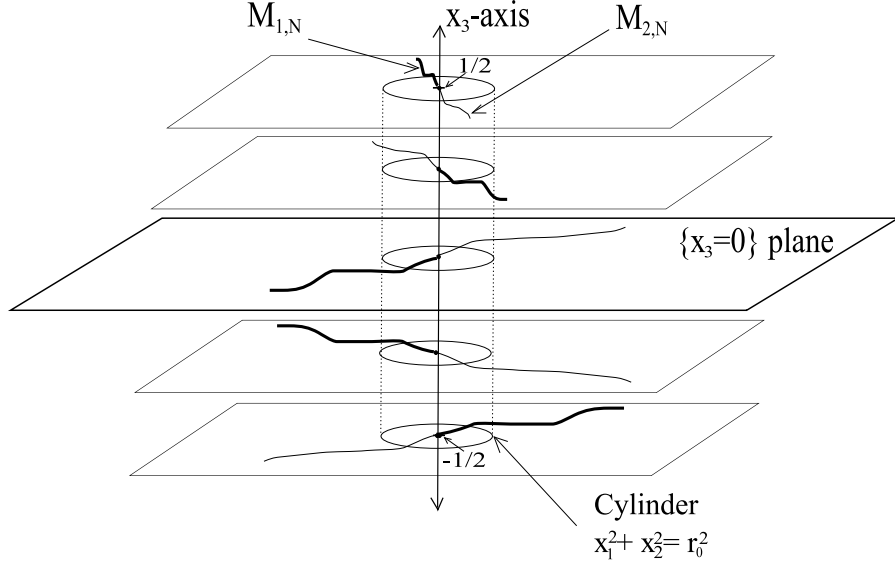
FIGURE 1. Horizontal slices of $M_N \setminus \{(0,0,t)|\,|t| \leq 1/2\} = M_{1,N} \cup M_{2,N}$ in Corollary 1

properties (a), (b) and (c). Theorem 2 further describes the convergence of this sequence to a limit lamination of the unit ball with singularities along the line segment $\{(0,0,t)\,|\,-1 \leq t \leq 0\}$.

**Theorem 2.** *There exists a sequence of compact embedded minimal disks* $0 \in \Sigma_N \subset B_1 \subset \mathbf{R}^3$ *with* $\partial\Sigma_N \subset \partial B_1$ *and each containing the vertical segment* $\{(0,0,t)\,|\,|t| \leq 1\} \subset \Sigma_N$ *with the following properties:*

(a) $\forall p \in \{(0,0,t)\,|\,-1 \leq t \leq 0\}$, $\lim_{N\to\infty} |A_{\Sigma_N}|^2(p) = \infty$.

(b) $\Sigma_N \setminus \{(0,0,t)|\,|t| \leq 1\} = \Sigma_{1,N} \cup \Sigma_{2,N}$ *for multi-valued graphs* $\Sigma_{1,N}$, $\Sigma_{2,N}$ *over the* $\{x_3 = 0\} \setminus \{0\}$ *punctured plane.*

(c) $\sup_N \sup_{\Sigma_N \setminus B_\delta} |A_{\Sigma_N}|^2 < \infty$ *for all* $\delta > 0$, *where* $B_\delta$ *is a* $\delta$-*neighborhood of* $\{(0,0,t)\,|\,-1 \leq t \leq 0\}$ .

*This sequence of compact embedded minimal disks converges to a minimal lamination of* $B_1 \setminus \{(0,0,t)\,|\,-1 \leq t \leq 0\}$ *consisting of a foliation by parallel planes of the lower hemisphere below* $\{x_3 = 0\}$ *and one leaf in the upper hemisphere,* $\Sigma$, *such that* $\Sigma \setminus \{x_3 - axis\} = \Sigma' \cup \Sigma''$, *where* $\Sigma'$ *and* $\Sigma''$ *are multi-valued graphs, each of which spirals into* $\{x_3 = 0\}$. *This limit lamination has removable singularities along the line segment* $\{(0,0,t)\,|\,-1 \leq t \leq 0\}$ *of the negative* $x_3$-*axis but the singularity at the origin cannot be removed.*

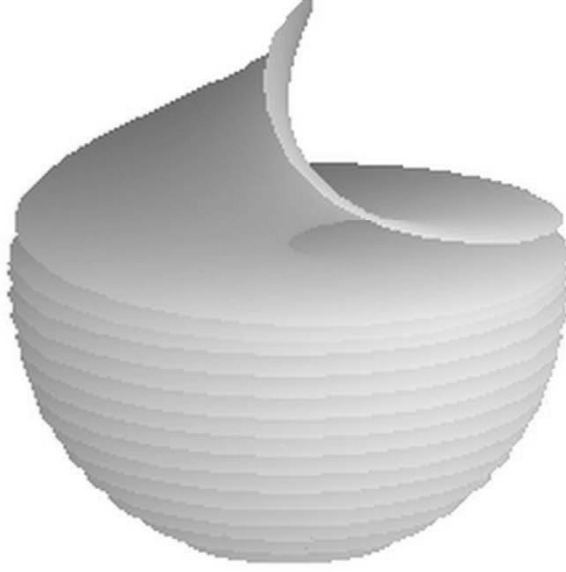Figure 2 shows a schematic picture of this limit lamination.

FIGURE 2. Schematic picture of the limit lamination in Theorem 2

## 1. NOTATION

In this paper we will use standard $(x_1, x_2, x_3)$ coordinates on $\mathbf{R}^3$ and $z = x + i\,y$ on $\mathbf{C}$. Given a function $f : \mathbf{C} \to \mathbf{C}^n$, we will use $\partial_x f$ and $\partial_y f$ to denote $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. Also, $\partial_z f = (\partial_x f - i\partial_y f)/2$.

For $p \in \mathbf{R}^3$ and $s > 0$, the ball of radius $s$ in $\mathbf{R}^3$ will be denoted by $B_s(p)$. $K_\Sigma$ is the sectional curvature of a smooth surface $\Sigma$. When $\Sigma$ is immersed in $\mathbf{R}^3$, we let its second fundamental form be $A_\Sigma$ (hence when $\Sigma$ is minimal, $|A_\Sigma|^2 = -2\,K_\Sigma$). When $\Sigma$ is oriented, we let $\mathbf{n}_\Sigma$ be the unit normal.

## 2. THE WEIERSTRASS REPRESENTATION

Given a meromorphic function $g(z)$ and a holomorphic one-form $\phi(z)$, defined on a domain $\Omega$, the Weierstrass representation of a conformal minimal immersion, $F : \Omega \to \mathbb{R}^3$, is given by (see [Os], Lemma 8.2):

$$(2.1) \quad F(z) = \operatorname{Re} \int_{\zeta \in \gamma_{z_0,z}} \left( \frac{1}{2}(g^{-1}(\zeta) - g(\zeta)), \frac{i}{2}(g^{-1}(\zeta) + g(\zeta)), 1 \right) \phi(\zeta)$$

, where $z_0$ is a fixed base point in $\Omega$ and the integration is taken along a path $\gamma_{z_0,z}$ from $z_0$ to $z$ in $\Omega$. In this paper we set the base point $z_0 = 0$, we choose $\phi$ such that it has no zeros and $g$ such that it has no poles or zeros in $\Omega$ and we choose $\Omega$ to be simply connected. These ensure that $F(z)$ does not depend on the choice of path $\gamma_{z_0,z}$ and that the differential of $F$, $dF$, is non-zero (and this ensures that $F$ is an immersion).

The unit normal $\mathbf{n}$ and the Gauss curvature $\mathbf{K}$ of this surface are (see [Os] sections 8, 9):

$$(2.2) \qquad \mathbf{n} = (2\mathrm{Re}g, 2\mathrm{Im}g, |g|^2 - 1)/(|g|^2 + 1)$$

$$(2.3) \qquad \mathbf{K} = - \left[ \frac{4|\partial_z g||g|}{|\phi|(1 + |g|^2)^2} \right]^2$$

We will need the following lemma, which follows immediately from the Weierstrass representation (2.1), that gives the differential of $F$:

**Lemma 1.** *If $F$ is given by (2.1) with $g(z) = e^{i\,(u(z)+iv(z))}$ and $\phi = dz$, then*

$$(2.4) \qquad \partial_x F = (\sinh v \, \cos u, \sinh v \, \sin u, 1) \,,$$

$$(2.5) \qquad \partial_y F = (\cosh v \, \sin u, -\cosh v \, \cos u, 0) \,.$$

## 3. Proof of the Main Theorems

We proceed to prove Theorem 1 by first constructing a family of minimal immersions $F_N$ with a specific choice of Weierstrass data $g(z) = e^{ih_N(z)}, \phi(z) = dz$, where $h_N(z) = u_N + iv_N$ and a corresponding domain $\Omega_N$ to obtain $F_N(z)$ from (2.1).

We first define $\partial_z h_N(z)$ because it is this derivative that will be essential in determining the curvature properties required of our sequence of embedded minimal disks. We let:

$$\partial_z h_N(z) = \frac{1}{2} \left[ \frac{1}{[z^2 + (\frac{1}{N})^2]^2} + \frac{1}{N} \sum_{k=1}^{N} \frac{1}{[(z + \frac{k}{N})^2 + (\frac{1}{N})^2]^2} \right],$$

for $N \geq 2$ on the domain $\Omega_N = \Omega_N^+ \cup \Omega_N^-$, where:

$$\Omega_N^+ = \left\{ (x,y) \,\middle|\, |y| \leq \frac{(x^2 + (\frac{1}{N})^2)^{5/4}}{4}, \, 0 < x \leq 1/2 \right\}$$

and

$$\Omega_N^- = \left\{ (x,y) \,|\, |y| \leq b_N, -1/2 \leq x \leq 0 \right\},$$

where $b_N = \dfrac{1}{4N^{5/2}}$. See Figure 3.

In this paper, we will denote the upper and lower boundary of $\Omega_N$ as $y_{x,N}$ and $-y_{x,N}$ respectively. That is, on $\Omega_N^+$, we set $y_{x,N} = \frac{(x^2+(\frac{1}{N})^2)^{5/4}}{4}$ and on $\Omega_N^-$, we set $y_{x,N} = b_N$.
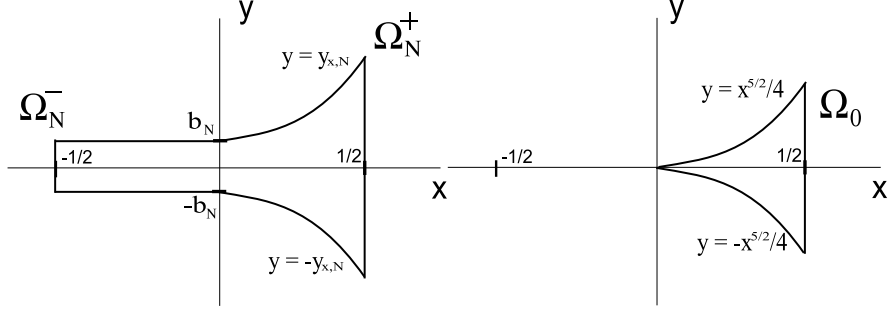
FIGURE 3. Diagram of the domain, $\Omega_N$

FIGURE 4. Diagram of the limit domain, $\Omega_0$

We note that for all $N$, $\partial_z h_N(z)$ is holomorphic on the domain $\Omega_N$ because the poles $\{\pm\frac{i}{N} - \frac{k}{N}\}$ for $0 \le k \le N$ lie outside the domain. Furthermore, these poles converge to the line segment $\{-1 \le x \le 0\}$ as $N \to \infty$.

Lemma 2 shows that there is a subsequence, $\{\partial_z h_{N_i}(z)\}$, that converges uniformly to a limit that we denote as:

$$\partial_z h(z) = \lim_{N_i \to \infty} \partial_z h_N(z)$$

on compact subsets of

$$\Omega_0 = \bigcap_N \Omega_N \setminus \{0\} = \left\{(x,y) \,\middle|\, |y| \le \frac{x^{5/2}}{4}, 0 < x \le 1/2\right\}$$

(See Figure 4). Therefore since for each $N$, $\partial_z h_N(z)$ is holomorphic on $\Omega_N$, we have that $\partial_z h(z)$, and hence also $h(z)$, is holomorphic on $\Omega_0$.

**Lemma 2.** *For $N \to \infty$, there is a subsequence $\{N_i\}$ such that $\{\partial_z h_{N_i}(z)\}$ converges uniformly on compact subsets of $\Omega_0$.*

*Proof.* For every compact subset, $K \subset \Omega_0$, $\exists r_K > 0$ such that $\forall z \in K$, $\forall N, k > 0$, $|(z + \frac{k}{N})^2 + (\frac{1}{N})^2| > r_K$, $|z^2 + (\frac{1}{N})^2| > r_K$

$$\implies |\partial_z h_N(z)| = \left| \frac{1}{2}\left[ \frac{1}{[z^2 + (\frac{1}{N})^2]^2} + \frac{1}{N}\sum_{k=1}^N \frac{1}{[(z + \frac{k}{N})^2 + (\frac{1}{N})^2]^2} \right]\right|$$

$< \frac{1}{2}\left[\frac{1}{r_K^2} + \frac{1}{r_K^2}\right] < \frac{1}{r_K^2}$

$\implies \{\partial_z h_N\}$ is a family of holomorphic functions , bounded on compact subsets of $\Omega_0$

$\implies$ by Montel's theorem, there is a subsequence that converges uniformly on compact subsets of $\Omega_0$ to a holomorphic limit. $\qquad\square$

Now, since for each $N$, $\partial_z h_N(z)$ is holomorphic on $\Omega_N$, we integrate to obtain our Weierstrass data:

$$(3.1) \quad h_N(z) = \frac{N^2}{4}\left[ N\arctan(Nz) + \frac{z}{z^2 + (\frac{1}{N})^2} \right.$$

$$\left. + \frac{1}{N}\sum_{k=1}^{N}\left( N\arctan(N(z+k/N)) + \frac{(z+k/N)}{(z+k/N)^2 + (\frac{1}{N})^2} \right) \right]$$

And since $h_N(z)$ is also holomorphic on $\Omega_N$, by the Cauchy-Riemann equations we have:

$$\partial_z h_N(z) = \partial_x u_N - i\partial_y u_N = \partial_y v_N + i\partial_x v_N$$

Therefore:

$$\partial_z h_N(z) = \frac{1}{2}\left[ \frac{1}{[z^2 + (\frac{1}{N})^2]^2} + \frac{1}{N}\sum_{k=1}^{N}\frac{1}{[(z+\frac{k}{N})^2 + (\frac{1}{N})^2]^2} \right]$$

$$= \frac{1}{2}\left[ \frac{(x^2 + (\frac{1}{N})^2 - y^2)^2 - 4x^2y^2 - 4ixy(x^2 + (\frac{1}{N})^2 - y^2)}{([x^2 + (\frac{1}{N})^2 - y^2]^2 + 4x^2y^2)^2} \right.$$

$$+ \frac{1}{N}\sum_{k=1}^{N}\left( \frac{((x+k/N)^2 + (\frac{1}{N})^2 - y^2)^2 - 4(x+k/N)^2y^2}{([(x+k/N)^2 + (\frac{1}{N})^2 - y^2]^2 + 4(x+k/N)^2y^2)^2} \right.$$

$$\left.\left. - \frac{4i(x+k/N)y((x+k/N)^2 + (\frac{1}{N})^2 - y^2)}{([(x+k/N)^2 + (\frac{1}{N})^2 - y^2]^2 + 4(x+k/N)^2y^2)^2} \right) \right]$$

$$(3.2) \quad \implies \partial_y u_N = \frac{1}{2}\left[ \frac{4xy(x^2 + (\frac{1}{N})^2 - y^2)}{([x^2 + (\frac{1}{N})^2 - y^2]^2 + 4x^2y^2)^2} \right.$$

$$\left. + \frac{1}{N}\sum_{k=1}^{N}\frac{4(x+k/N)y((x+k/N)^2 + (\frac{1}{N})^2 - y^2)}{([(x+k/N)^2 + (\frac{1}{N})^2 - y^2]^2 + 4(x+k/N)^2y^2)^2} \right]$$

And

$$(3.3) \quad \partial_y v_N = \frac{1}{2}\left[ \frac{(x^2 + (\frac{1}{N})^2 - y^2)^2 - 4x^2y^2}{([x^2 + (\frac{1}{N})^2 - y^2]^2 + 4x^2y^2)^2} \right.$$

$$\left. + \frac{1}{N}\sum_{k=1}^{N}\frac{((x+k/N)^2 + (\frac{1}{N})^2 - y^2)^2 - 4(x+k/N)^2y^2}{([(x+k/N)^2 + (\frac{1}{N})^2 - y^2]^2 + 4(x+k/N)^2y^2)^2} \right]$$

Now the main difficulty we encounter in the proof of Theorem 1 is showing that the immersions $F_N : \Omega_N \to \mathbf{R}^3$ are in fact embeddings.

The next Lemma gives this embeddedness result.

**Lemma 3.** *There exists $r_0 > 0$ (independent of $N$) such that $\forall (x, y) \in \Omega_N$,*

(3.4)        $x_3(F_N(x, y)) = x$.

(3.5)        *The curve $F_N(x, \cdot) : [-y_{x,N}, y_{x,N}] \to \{x_3 = x\}$*
             *is a graph in the $\{x_3 = x\}$ plane.*

(3.6)        $|F_N(x, \pm y_{x,N}) - F_N(x, 0)| > r_0$ *for all $N$.*

(3.7)        *In fact, for $x \leq 0$, $|F_N(x, \pm y_{x,N}) - F_N(x, 0)| \to \infty$ as $N \to \infty$*

In this Lemma, (3.4) shows that the horizontal slice of the image, $F_N(\Omega_N) \cap \{x_3 = t\}$, is the image of the vertical line $\{x = t\}$ in the domain $(\Omega_N)$. (3.5) shows that the image $F_N(\{x = t\} \cap \Omega_N)$ is a graph in the $\{x_3 = t\}$ plane over a line segment in that plane (see Figure 5). Together, these imply embeddedness. Also, (3.6) shows that there is some $r_0$ such that the boundary of the graph in (3.5) lies outside a circle $B_{r_0}(F_N(t, 0))$ for all $N$. And (3.7) shows that for all $x \leq 0$ (i.e. in the part of the image $F_N(\Omega_N)$ below the $\{x_3 = 0\}$ plane), these boundaries of the graph in (3.5) actually go to infinity as $N \to \infty$.
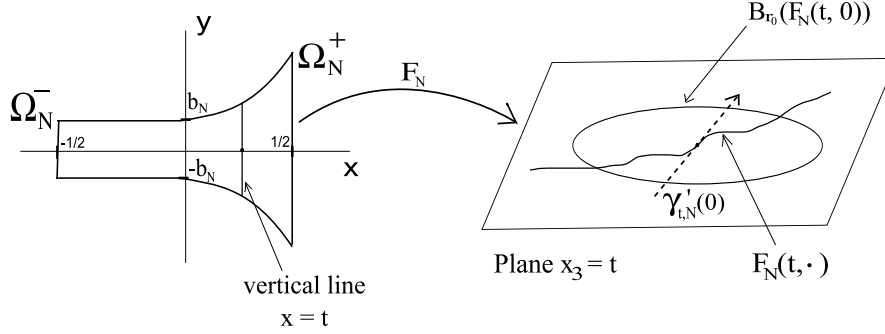


FIGURE 5. A horizontal slice of $F(\Omega_N)$ in Lemma 3

*Proof.* Since $z_0 = 0$ and the height differential is $\phi = dz$, (3.4) follows immediately from (2.1).

Now we prove (3.5) first for $0 < x \leq \frac{1}{2}$ (i.e. on $\Omega_N^+$) and then for $-\frac{1}{2} \leq x \leq 0$ (i.e. on $\Omega_N^-$)

3.1. **Proof of (3.5) on $\Omega_N^+$.** We first note that on

$$\Omega_N^+ = \left\{ (x, y) \,\middle|\, |y| \leq \frac{(x^2 + (\frac{1}{N})^2)^{5/4}}{4}, \, 0 < x \leq 1/2 \right\},$$

$$(3.8) \qquad y^2 \leq \frac{(x^2 + (\frac{1}{N})^2)^{5/2}}{16} \leq \frac{x^2 + (\frac{1}{N})^2}{16} \leq \frac{(x + \frac{k}{N})^2 + (\frac{1}{N})^2}{16};$$

$$(3.9) \qquad y^2 \leq \frac{(x^2 + (\frac{1}{N})^2)^{5/2}}{16} \leq \frac{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^{5/2}}{16}$$

for all $k \geq 1$ since $0 < x \leq 1/2$ and $N \geq 2 \implies (\frac{1}{N}) \leq 1/2$

Now using this in (3.2),

$$|\partial_y u_N| \leq \frac{1}{2} \left[ \frac{4x|y||x^2 + (\frac{1}{N})^2 - y^2|}{([x^2 + (\frac{1}{N})^2 - y^2]^2 + 4x^2 y^2)^2} \right.$$

$$\left. + \frac{1}{N} \sum_{k=1}^{N} \frac{4(x + k/N)|y||(x + k/N)^2 + (\frac{1}{N})^2 - y^2|}{([(x + k/N)^2 + (\frac{1}{N})^2 - y^2]^2 + 4(x + k/N)^2 y^2)^2} \right]$$

$$\leq \frac{1}{2} \left[ \frac{4x|y|(x^2 + (\frac{1}{N})^2)}{([\frac{15}{16}(x^2 + (\frac{1}{N})^2)]^2)^2} \right.$$

$$\left. + \frac{1}{N} \sum_{k=1}^{N} \frac{4(x + k/N)|y|((x + k/N)^2 + (\frac{1}{N})^2)}{([\frac{15}{16}((x + k/N)^2 + (\frac{1}{N})^2)]^2)^2} \right]$$

$$= 2 \left( \frac{16}{15} \right)^4 |y| \left[ \frac{x}{(x^2 + (\frac{1}{N})^2)^3} + \frac{1}{N} \sum_{k=1}^{N} \frac{(x + \frac{k}{N})}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^3} \right]$$

We set $y_{x,N} = \dfrac{(x^2 + (\frac{1}{N})^2)^{5/4}}{4}$

$$\implies \max_{|y| \leq y_{x,N}} |u_N(x, y) - u_N(x, 0)|$$

$$\leq 2 \left( \frac{16}{15} \right)^4 \left( \int_0^{y_{x,N}} t \, dt \right) \left[ \frac{x}{(x^2 + (\frac{1}{N})^2)^3} + \frac{1}{N} \sum_{k=1}^{N} \frac{(x + \frac{k}{N})}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^3} \right]$$

$$= \left( \frac{16}{15} \right)^4 y_{x,N}^2 \left[ \frac{x}{(x^2 + (\frac{1}{N})^2)^3} + \frac{1}{N} \sum_{k=1}^{N} \frac{(x + \frac{k}{N})}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^3} \right]$$

$$= \left( \frac{16}{15} \right)^4 \frac{(x^2 + (\frac{1}{N})^2)^{5/2}}{16} \left[ \frac{x}{(x^2 + (\frac{1}{N})^2)^3} + \frac{1}{N} \sum_{k=1}^{N} \frac{(x + \frac{k}{N})}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^3} \right]$$

$$= \left( \frac{16}{15} \right)^4 \frac{1}{16} \left[ \frac{x(x^2 + (\frac{1}{N})^2)^{5/2}}{(x^2 + (\frac{1}{N})^2)^3} + \frac{1}{N} \sum_{k=1}^{N} \frac{(x + \frac{k}{N})((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^{5/2}}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^3} \right]$$

$$= \left( \frac{16}{15} \right)^4 \frac{1}{16} \left[ \frac{x}{(x^2 + (\frac{1}{N})^2)^{\frac{1}{2}}} + \frac{1}{N} \sum_{k=1}^{N} \frac{(x + \frac{k}{N})}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^{\frac{1}{2}}} \right]$$

$$(3.10) \qquad \leq \left( \frac{16}{15} \right)^4 \frac{1}{16} [1 + 1] < 1$$

We set $\gamma_{x,N}(y) = F_N(x,y) \implies \gamma'_{x,N}(y) = \partial_y F_N(x,y)$. We note that $v_N(x,0) = 0$ from (3.1) and that $\cos(1) > 1/2$.

Therefore (2.5) gives:

(3.11)
$$\langle \gamma'_{x,N}(y), \gamma'_{x,N}(0) \rangle = \cosh v_N(x,y) \cos(u_N(x,y) - u_N(x,0)) > \frac{1}{2} \cosh v_N(x,y)$$

Hence, by (3.11), the angle between $\gamma'_{x,N}(y)$ and $\gamma'_{x,N}(0)$ is always less than $\pi/2$. This gives us (3.5) for all $0 < x \leq \frac{1}{2}$ (i.e. on $\Omega_N^+$)

Furthermore, this result holds uniformly in $N$ because of the uniform bound in (3.10).

### 3.2. Proof of (3.5) on $\Omega_N^-$.

We recall that $\Omega_N^- = \{(x,y) \,|\, |y| \leq b_N, -1/2 \leq x \leq 0\}$, where $b_N = \dfrac{1}{4N^{5/2}}$

Now we note that on $\Omega_N^-$, for $N \geq 2$,

(3.12)
$$y^2 \leq b_N^2 = \frac{1}{16N^5} \leq \frac{(x^2 + (\frac{1}{N})^2)^{5/2}}{16}$$

and

(3.13)
$$y^2 \leq b_N^2 = \frac{1}{16N^5} \leq \frac{((x + k/N)^2 + (\frac{1}{N})^2)^{5/2}}{16}$$

Also,

(3.14)
$$y^2 \leq b_N^2 = \frac{1}{16N^5} < \frac{x^2 + (\frac{1}{N})^2}{16}$$

and

(3.15)
$$y^2 \leq b_N^2 = \frac{1}{16N^5} < \frac{(x + k/N)^2 + (\frac{1}{N})^2}{16}$$

Now using these inequalities in (3.2),

$$|\partial_y u_N| \leq \frac{1}{2} \left[ \frac{4x|y||x^2 + (\frac{1}{N})^2 - y^2|}{([x^2 + (\frac{1}{N})^2 - y^2]^2 + 4x^2y^2)^2} \right.$$

$$+ \frac{1}{N} \sum_{k=1}^{N} \left. \frac{4(x + k/N)|y||(x + k/N)^2 + (\frac{1}{N})^2 - y^2|}{([(x + k/N)^2 + (\frac{1}{N})^2 - y^2]^2 + 4(x + k/N)^2 y^2)^2} \right]$$

$$\leq \frac{1}{2} \left[ \frac{4x|y|(x^2 + (\frac{1}{N})^2)}{([\frac{15}{16}(x^2 + (\frac{1}{N})^2)]^2)^2} + \frac{1}{N} \sum_{k=1}^{N} \frac{4(x + k/N)|y|((x + k/N)^2 + (\frac{1}{N})^2)}{([\frac{15}{16}((x + k/N)^2 + (\frac{1}{N})^2)]^2)^2} \right]$$

$$= 2 \left( \frac{16}{15} \right)^4 |y| \left[ \frac{x}{(x^2 + (\frac{1}{N})^2)^3} + \frac{1}{N} \sum_{k=1}^{N} \frac{(x + \frac{k}{N})}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^3} \right]$$

We recall that $y_{x,N} = b_N$

$$\implies \max_{|y| \leq y_{x,N}} |u_N(x,y) - u_N(x,0)|$$

$$\leq 2 \left(\frac{16}{15}\right)^4 \left(\int_0^{y_{x,N}} t\, dt\right) \left[\frac{x}{(x^2 + (\frac{1}{N})^2)^3} + \frac{1}{N} \sum_{k=1}^N \frac{(x + \frac{k}{N})}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^3}\right]$$

$$= \left(\frac{16}{15}\right)^4 y_{x,N}^2 \left[\frac{x}{(x^2 + (\frac{1}{N})^2)^3} + \frac{1}{N} \sum_{k=1}^N \frac{(x + \frac{k}{N})}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^3}\right]$$

$$= \left(\frac{16}{15}\right)^4 b_N^2 \left[\frac{x}{(x^2 + (\frac{1}{N})^2)^3} + \frac{1}{N} \sum_{k=1}^N \frac{(x + \frac{k}{N})}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^3}\right]$$

$$\leq \left(\frac{16}{15}\right)^4 \frac{1}{16} \left[\frac{x(x^2 + (\frac{1}{N})^2)^{5/2}}{(x^2 + (\frac{1}{N})^2)^3} + \frac{1}{N} \sum_{k=1}^N \frac{(x + \frac{k}{N})((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^{5/2}}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^3}\right]$$

$$= \left(\frac{16}{15}\right)^4 \frac{1}{16} \left[\frac{x}{(x^2 + (\frac{1}{N})^2)^{\frac{1}{2}}} + \frac{1}{N} \sum_{k=1}^N \frac{(x + \frac{k}{N})}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^{\frac{1}{2}}}\right]$$

$$(3.16) \qquad\qquad \leq \left(\frac{16}{15}\right)^4 \frac{1}{16}[1 + 1] < 1$$

And now we use the same argument we used to show (3.5) on $\Omega_N^+$. We set $\gamma_{x,N}(y) = F_N(x,y)$. We note that $v_N(x,0) = 0$ and $\cos(1) > 1/2$.

Therefore (2.5) gives:

$$(3.17) \qquad \langle \gamma'_{x,N}(y), \gamma'_{x,N}(0)\rangle = \cosh v_N(x,y)\cos(u_N(x,y) - u_N(x,0))$$

$$> \frac{1}{2}\cosh v_N(x,y)$$

Hence, by (3.17), the angle between $\gamma'_{x,N}(y)$ and $\gamma'_{x,N}(0)$ is always less than $\pi/2$. This gives us (3.5) for all $-1/2 \leq x \leq 0$ (i.e. on $\Omega_N^-$)

Furthermore, this result holds uniformly in $N$ because of the uniform bound in (3.16). Now we prove (3.6) first for $0 < x \leq \frac{1}{2}$ (i.e. on $\Omega_N^+$) and then for $-\frac{1}{2} \leq x \leq 0$ (i.e. on $\Omega_N^-$).

### 3.3. **Proof of (3.6) on $\Omega_N^+$.** We recall that

$$\Omega_N^+ = \left\{(x,y) \,\Big|\, |y| \leq \frac{(x^2 + (\frac{1}{N})^2)^{5/4}}{4}, \, 0 < x \leq 1/2\right\},$$

From (3.3), and by using (3.8) we have:

$$\partial_y v_N = \frac{1}{2}\left[\frac{(x^2 + (\frac{1}{N})^2 - y^2)^2 - 4x^2 y^2}{([x^2 + (\frac{1}{N})^2 - y^2]^2 + 4x^2 y^2)^2}\right.$$

$$\left. + \frac{1}{N}\sum_{k=1}^{N}\frac{((x + \frac{k}{N})^2 + (\frac{1}{N})^2 - y^2)^2 - 4(x + \frac{k}{N})^2 y^2}{([(x + \frac{k}{N})^2 + (\frac{1}{N})^2 - y^2]^2 + 4(x + \frac{k}{N})^2 y^2)^2}\right]$$

$$\geq \frac{1}{2}\left[\frac{(\frac{15}{16}(x^2 + (\frac{1}{N})^2))^2 - 4(x^2 + (\frac{1}{N})^2)\frac{1}{16}(x^2 + (\frac{1}{N})^2)}{([x^2 + (\frac{1}{N})^2]^2 + 4(x^2 + (\frac{1}{N})^2)\frac{1}{16}(x^2 + (\frac{1}{N})^2))^2} + \frac{1}{N}\sum_{k=1}^{N}\right.$$

$$\left(\frac{(\frac{15}{16}((x + \frac{k}{N})^2 + (\frac{1}{N})^2))^2}{([(x + \frac{k}{N})^2 + (\frac{1}{N})^2]^2 + 4((x + \frac{k}{N})^2 + (\frac{1}{N})^2)\frac{1}{16}((x + \frac{k}{N})^2 + (\frac{1}{N})^2))^2}\right.$$

$$\left.\left.- \frac{4((x + \frac{k}{N})^2 + (\frac{1}{N})^2)\frac{1}{16}((x + \frac{k}{N})^2 + (\frac{1}{N})^2)}{([(x + \frac{k}{N})^2 + (\frac{1}{N})^2]^2 + 4((x + \frac{k}{N})^2 + (\frac{1}{N})^2)\frac{1}{16}((x + \frac{k}{N})^2 + (\frac{1}{N})^2))^2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\frac{161}{256}(x^2 + (\frac{1}{N})^2)^2}{\frac{25}{16}[x^2 + (\frac{1}{N})^2]^4} + \frac{1}{N}\sum_{k=1}^{N}\frac{\frac{161}{256}((x + k/N)^2 + (\frac{1}{N})^2)^2}{\frac{25}{16}[(x + \frac{k}{N})^2 + (\frac{1}{N})^2]^4}\right]$$

$$= \frac{161}{800}\left[\frac{1}{[x^2 + (\frac{1}{N})^2]^2} + \frac{1}{N}\sum_{k=1}^{N}\frac{1}{[(x + \frac{k}{N})^2 + (\frac{1}{N})^2]^2}\right]$$

$$(3.18) \qquad\qquad \implies \partial_y v_N \geq \frac{161}{800[x^2 + (\frac{1}{N})^2]^2}$$

We recall that $y_{x,N} = \dfrac{(x^2 + (\frac{1}{N})^2)^{5/4}}{4}$

$$\implies \min_{y_{x,N}/2 \leq |y| \leq y_{x,N}} |v_N(x,y)| \geq \int_0^{y_{x,N}/2}\frac{161}{800[x^2 + (\frac{1}{N})^2]^2}\, dt$$

$$= \frac{161}{6400[x^2 + (\frac{1}{N})^2]^{3/4}}$$

From (3.11), we have $\langle \gamma'_{x,N}(y), \gamma'_{x,N}(0)\rangle > \frac{1}{2}\cosh v_N(x,y)$.

Integrating this gives $\langle \gamma_{x,N}(y_{x,N}) - \gamma_{x,N}(0), \gamma'_{x,N}(0)\rangle$

$$> \int_{y_{x,N}/2}^{y_{x,N}}\frac{1}{2}\cosh(v_N(x,y))\, dy$$

$$\geq \frac{1}{2}\frac{(x^2 + (\frac{1}{N})^2)^{5/4}}{8}\cosh\left(\frac{161}{6400[x^2 + (\frac{1}{N})^2]^{3/4}}\right)$$

Now, since $|\gamma'_{x,N}(0)| = \cosh v_N(x,0) = 1$ and

$$\lim_{s \to 0} s^{5/4}\cosh\left(\frac{161}{6400 s^{3/4}}\right) = \infty,$$ this result and the analog for

$\gamma_{x,N}(-y_{x,N})$ give our result on $\Omega_N^+$, (3.6), that

$$(3.19) \qquad \forall x \in (0, 1/2), \left|F_N(x, \pm\frac{(x^2 + (\frac{1}{N})^2)^{5/4}}{4}) - F_N(x, 0)\right| > r_1$$

for some $r_1 > 0$ (independent of $N$) and all $N \geq 2$.

3.4. **Proof of (3.6) on $\Omega_N^-$.** We recall that $\Omega_N^- = \{(x,y) \,|\, |y| \leq b_N, -1/2 \leq x \leq 0\}$, where $b_N = \dfrac{1}{4N^{5/2}}$

Now on $\Omega_N^-$, from (3.3), and by using (3.14) and (3.15), we have:

$$\partial_y v_N = \frac{1}{2} \left[ \frac{(x^2 + (\frac{1}{N})^2 - y^2)^2 - 4x^2 y^2}{([x^2 + (\frac{1}{N})^2 - y^2]^2 + 4x^2 y^2)^2} \right.$$

$$\left. + \frac{1}{N} \sum_{k=1}^{N} \frac{((x + \frac{k}{N})^2 + (\frac{1}{N})^2 - y^2)^2 - 4(x + \frac{k}{N})^2 y^2}{([(x + \frac{k}{N})^2 + (\frac{1}{N})^2 - y^2]^2 + 4(x + \frac{k}{N})^2 y^2)^2} \right]$$

$$\geq \frac{1}{2} \left[ \frac{(\frac{15}{16}(x^2 + (\frac{1}{N})^2))^2 - 4(x^2 + (\frac{1}{N})^2)\frac{1}{16}(x^2 + (\frac{1}{N})^2)}{([x^2 + (\frac{1}{N})^2]^2 + 4(x^2 + (\frac{1}{N})^2)\frac{1}{16}(x^2 + (\frac{1}{N})^2))^2} + \frac{1}{N} \sum_{k=1}^{N} \right.$$

$$\left( \frac{(\frac{15}{16}((x + \frac{k}{N})^2 + (\frac{1}{N})^2))^2}{([(x + \frac{k}{N})^2 + (\frac{1}{N})^2]^2 + 4((x + \frac{k}{N})^2 + (\frac{1}{N})^2)\frac{1}{16}((x + \frac{k}{N})^2 + (\frac{1}{N})^2))^2} \right.$$

$$\left. \left. - \frac{4((x + \frac{k}{N})^2 + (\frac{1}{N})^2)\frac{1}{16}((x + \frac{k}{N})^2 + (\frac{1}{N})^2)}{([(x + \frac{k}{N})^2 + (\frac{1}{N})^2]^2 + 4((x + \frac{k}{N})^2 + (\frac{1}{N})^2)\frac{1}{16}((x + \frac{k}{N})^2 + (\frac{1}{N})^2))^2} \right) \right]$$

$$= \frac{1}{2} \left[ \frac{\frac{161}{256}(x^2 + (\frac{1}{N})^2)^2}{\frac{25}{16}[x^2 + (\frac{1}{N})^2]^4} + \frac{1}{N} \sum_{k=1}^{N} \frac{\frac{161}{256}((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^2}{\frac{25}{16}[(x + \frac{k}{N})^2 + (\frac{1}{N})^2]^4} \right]$$

$$= \frac{161}{800} \left[ \frac{1}{[x^2 + (\frac{1}{N})^2]^2} + \frac{1}{N} \sum_{k=1}^{N} \frac{1}{[(x + \frac{k}{N})^2 + (\frac{1}{N})^2]^2} \right]$$

$$\geq \frac{161}{800} \left[ \frac{1}{N} \sum_{k=1}^{N} \frac{1}{[(x + \frac{k}{N})^2 + (\frac{1}{N})^2]^2} \right]$$

Now $\forall x \in [-\frac{1}{2}, 0]$, $\forall N \geq 2$, $\exists t_x \in \mathbb{Z}$, $1 \leq t_x \leq N$ s.t. $-\frac{t_x}{N} < x \leq -\frac{t_x - 1}{N}$
$\implies x + \frac{t_x}{N} \leq \frac{1}{N}$
Hence $\forall \, -\frac{1}{2} \leq x \leq 0$,

$$\partial_y v_N \geq \frac{161}{800} \frac{1}{N} \frac{1}{[(x + \frac{t_x}{N})^2 + (\frac{1}{N})^2]^2} \geq \frac{161}{800} \frac{1}{N} \frac{1}{[(\frac{1}{N})^2 + (\frac{1}{N})^2]^2}$$

$$(3.20) \qquad\qquad \implies \partial_y v_N \geq \frac{161}{3200} N^3$$

$$\implies \min_{y_{x,N}/2 \leq |y| \leq y_{x,N}} |v_N(x,y)| \geq \int_0^{y_{x,N}/2} \frac{161}{3200} N^3 \, dt$$

$$= \frac{161}{3200} N^3 \frac{y_{x,N}}{2} = \frac{161}{6400} N^3 b_N$$

$$= \frac{161}{6400}N^3 \left( \frac{1}{4N^{5/2}} \right) = \frac{161}{25600}N^{1/2}$$

From (3.11), we have $\langle \gamma'_{x,N}(y), \gamma'_{x,N}(0) \rangle > \frac{1}{2} \cosh v_N(x,y)$

Integrating this gives

$$\langle \gamma_{x,N}(y_{x,N}) - \gamma_{x,N}(0), \gamma'_{x,N}(0) \rangle > \int_{y_{x,N}/2}^{y_{x,N}} \frac{1}{2} \cosh(v_N(x,y)) \, dy$$

$$\geq \frac{1}{2} \frac{b_N}{2} \cosh \left( \frac{161}{25600} N^{1/2} \right)$$

$$(3.21) \quad \implies \langle \gamma_{x,N}(y_{x,N}) - \gamma_{x,N}(0), \gamma'_{x,N}(0) \rangle > \frac{1}{16N^{5/2}} \cosh \left( \frac{161}{25600} N^{1/2} \right)$$

Now, since $|\gamma'_{x,N}(0)| = \cosh v_N(x,0) = 1$ and

$\lim_{N \to \infty} \frac{1}{16N^{5/2}} \cosh \left( \frac{161}{25600} N^{1/2} \right) = \infty$, this result and the analog for

$\gamma_{x,N}(-y_{x,N})$ give our result on $\Omega_N^-$ , (3.6), that

$$(3.22) \qquad \forall x \in [-1/2, 0], \, |F_N(x, \pm b_N) - F_N(x, 0)| > r_2$$

for some $r_2 > 0$ (independent of $N$) and all $N \geq 2$.

Hence, by choosing $r_0 = \min\{r_1, r_2\}$ given by (3.19) and (3.22), we have (3.6).

Also by (3.21) we have the result (3.7) that for $x \leq 0$, $|F_N(x, \pm b_N) - F_N(x, 0)| \to \infty$ as $N \to \infty$.

$\square$

Now we will prove the following corollary that gives us the embeddings $F_N$ that we will use in the proof of Theorem 1.

**Corollary 1.** *Let $r_0$ be given by (3.6).*
  (i) *$F_N$ is an embedding and $F_N(\Omega_N) \subset \mathbf{R}^2 \times [-1/2, 1/2] \subset \mathbf{R}^3$.*
  (ii) *$F_N(t, 0) = (0, 0, t)$ for $|t| \leq 1/2$.*
  (iii) *$F_N(\Omega_N) \setminus \{(0, 0, t) \mid |t| \leq 1/2\} = M_{1,N} \cup M_{2,N}$ for multi-valued graphs $M_{1,N}, M_{2,N}$ over the $\{x_3 = 0\} \setminus \{0\}$ punctured plane.*

*Proof.* (i) follows from (3.4) and (3.5).

We obtain (ii) by integrating (2.4) with respect to $x$, using the fact that $v_N(x, 0)$ is identically 0, and the fact that $F(0, 0) = (0, 0, 0)$ because of our choice of $z_0 = 0$ in (2.1). From (2.2), $F_N$ is "vertical" (i.e. $\langle \mathbf{n}, (0, 0, 1) \rangle = 0$) exactly when $|g_N| = 1$. But since $g_N(z) = e^{i(u_N(z) + iv_N(z))}$, $|g_N(x, y)| = 1 \iff v_N(x, y) = 0$. Now for $x > 0$, by (3.18) and since $v_N(x, 0) = 0$, we have $|v_N(x, y)| \geq \frac{161|y|}{800[x^2 + (\frac{1}{N})^2]^2}$. Similarly, for $x \leq 0$, by (3.20) and since $v_N(x, 0) = 0$, we have $|v_N(x, y)| \geq \frac{161|y|}{3200} N^3$. Hence, for all $x$, $v_N(x, y) = 0 \iff y = 0$ and therefore $\langle \mathbf{n}, (0, 0, 1) \rangle = 0 \iff y = 0$.

Therefore by Corollary 1, (ii), the image of $F_N$ is graphical away from the $x_3$-axis, giving us (iii).

$\square$

### 3.5. Proof of Theorem 1.

Corollary 1 gives us a sequence of minimal embeddings $F_N : \Omega_N \to \mathbf{R}^2 \times [-1/2, 1/2] \subset \mathbf{R}^3$ with $F_N(t, 0) = (0, 0, t)$ for $|t| \leq 1/2$.

We let $M_N = F_N(\Omega_N)$.

#### 3.5.1. *Proof of Theorem 1 (a).* By using (2.3) with our Weierstrass data, $g(z) = e^{i(u(z)+iv(z))}$ and $\phi = dz$, we have that the curvature of $F_N$ is given by

$$(3.23) \qquad K_N(z) = \frac{-|\partial_z h_N|^2}{\cosh^4 v_N}$$

Therefore, if $|\partial_z h_N| \to \infty$ and for some constant $M > 0, \cosh^4 v_N < M$, then $K_N \to \infty$

Let $z \in [-1/2, 0]$. $\forall N \geq 2, \exists t_z \in \mathbb{Z}, 1 \leq t_z \leq N$ s.t. $-\dfrac{t_z}{N} < z \leq -\dfrac{t_z - 1}{N}$

$\implies z + \dfrac{t_z}{N} \leq \dfrac{1}{N}$

$$\implies \partial_z h_N(z) = \frac{1}{2} \left[ \frac{1}{[z^2 + (\frac{1}{N})^2]^2} + \frac{1}{N} \sum_{k=1}^{N} \frac{1}{[(z + \frac{k}{N})^2 + (\frac{1}{N})^2]^2} \right]$$

$\geq \dfrac{1}{2N} \dfrac{1}{[(z + \frac{t_z}{N})^2 + (\frac{1}{N})^2]^2} \geq \dfrac{1}{2N} \dfrac{1}{[(\frac{1}{N})^2 + (\frac{1}{N})^2]^2} = N^3/8 \implies \lim\limits_{N \to \infty} \partial_z h_N(z) = \infty$

Now, we also note that $\forall x, -1/2 \leq x \leq 1/2, v_N(x, 0) = 0$.

$\implies \cosh^4(v_N(x, 0)) = 1$

Hence, we have curvature blowing up at all points of the line segment, $[-1/2, 0] \subset \Omega_N$.

This gives us that $\forall p \in \{(0, 0, t) \mid -1/2 \leq t \leq 0\}, \lim\limits_{N \to \infty} |A_{M_N}|^2(p) = \infty$.

#### 3.5.2. *Proof of Theorem 1 (b) and (c).* Theorem 1 (b) follows immediately from Corollary 1(iii).

To prove Theorem 1 (c) we fix $\delta > 0$, and let $B_\delta$ be a $\delta$-neighborhood of $\{(0, 0, t) \mid -1/2 \leq t \leq 0\}$ that is cylindrically shaped (shown in Figure 6).

Then $\forall N, \forall p = (x_1, x_2, x_3) \in M_N$, where $x_3 > \delta$ (i.e. for all points of $M_N$ that are more than a $\delta$ distance above the $x_3 = 0$ plane), by (3.4), $x = x_3 > \delta$, where $(x, y) \in \Omega_N^+$ such that $F_N(x, y) = (x_1, x_2, x_3)$.
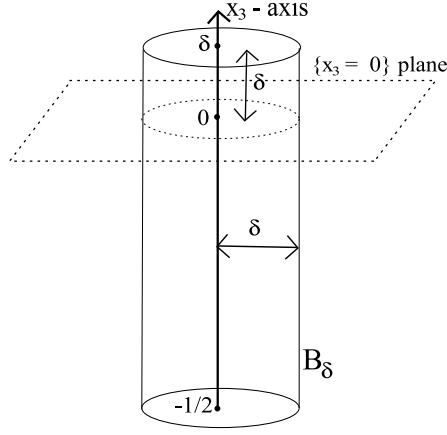
FIGURE 6. Diagram of $B_\delta$, the $\delta$-neighborhood of $\{(0, 0, t) \mid -1/2 \leq t \leq 0\}$ in Theorem 1

Hence, since on $\Omega_N^+$, $y^2 \leq \dfrac{x^2 + 1/N^2}{16} \leq \dfrac{(x + k/N)^2 + 1/N^2}{16}$,

$|z^2 + (\dfrac{1}{N})^2|^2 = |(x + iy)^2 + (1/N)^2|^2 = [x^2 + (1/N)^2 - y^2]^2 + 4x^2y^2$

$\geq (15/16)^2[x^2 + (1/N)^2]^2 + 4x^2y^2 \geq (15/16)^2 x^4 > (15/16)^2 \delta^4$

Similarly, $|(z + k/N)^2 + (\dfrac{1}{N})^2|^2 = |(x + k/N + iy)^2 + (1/N)^2|^2 = [(x + k/N)^2 + (1/N)^2 - y^2]^2 + 4(x + k/N)^2 y^2 \geq (15/16)^2[(x + k/N)^2 + (1/N)^2]^2 + 4(x + k/N)^2 y^2 \geq (15/16)^2 x^4 > (15/16)^2 \delta^4$

$$\implies |\partial_z h_N(z)| = \left| \frac{1}{2} \left[ \frac{1}{[z^2 + (\frac{1}{N})^2]^2} + \frac{1}{N} \sum_{k=1}^{N} \frac{1}{[(z + \frac{k}{N})^2 + (\frac{1}{N})^2]^2} \right] \right|$$

$$< \frac{1}{2} \left[ \frac{1}{(15/16)^2 \delta^4} + \frac{1}{(15/16)^2 \delta^4} \right] < \frac{1}{(15/16)^2 \delta^4}$$

This uniform bound of $|\partial_z h_N(z)|$ gives the curvature bound for all points of $M_N$ that are at least a distance $\delta$ above the $x_3 = 0$ plane by (3.23). Let this curvature bound be $C_\delta^{(1)}$.

Now, at all points $p \in M_N$ that are at least a distance $\delta$ away from the $x_3-$axis (i.e. outside a cylinder about the $x_3-$axis of radius $\delta$), Heinz's curvature estimate for graphs (11.7 in [Os]) applied to components of $M_N$ over disks of radius $\delta/2$ in the $\{x_3 = 0\}$ plane, which are guaranteed to be graphs over the $\{x_3 = 0\}$ plane by Theorem 1(b), gives a uniform curvature bound, $C_\delta^{(2)}$.

Hence, $C_\delta = \max\{C_\delta^{(1)}, C_\delta^{(2)}\}$ is the uniform curvature bound in Theorem 1(c).

Theorem 1(d) follows from (3.6) and (3.7).

3.6. **Proof of Theorem 2.** We will need the following lemma that gives us convergence.

**Lemma 4.** *Consider the sequence of embedded minimal disks $\{M_N\}$ given by Theorem 1. Let $W = \mathcal{C} \cup H$ where $\mathcal{C}$ is the cylinder $\{(x_1, x_2, x_3)|\, x_1^2 + x_2^2 \leq r_0^2, -1/2 < x_3 < 1/2\}$ with $r_0$ determined by Theorem 1(d) and $H = \mathbf{R}^2 \times [-1/2, 0]$ (a horizontal block of the half space below the $\{x_3 = 0\}$ axis).*

   (a) *$\{M_N\}$, as a sequence of minimal laminations, has a subsequence that converges to a limit lamination on compact subsets of $W$ away from $\{(0, 0, t)|\, -1/2 \leq t \leq 0\}$ in the $C^\alpha$ topology for any $\alpha < 1$.*

   (b) *This subsequence of embedded minimal disks has a further subsequence $\{M_{N_i}\}$ such that the leaves converge uniformly in the $C^k$ topology for all $k$.*

*Proof.* To prove Lemma 4(a) we cover compact subsets $K$ of $W \setminus \{(0, 0, t)|\, -1/2 \leq t \leq 0\}$ with sufficiently small balls $B_{r_K}$ (with radius $r_K$ depending on the compact subset) such that the covering does not intersect $\{(0, 0, t)|\, |t| \leq 1/2\}$. For each $K$, we take $N$ in the sequence $\{M_N\}$ to be large enough (i.e. $N \geq N_K$ for some $N_K$ depending on $K$) such that $\partial M_N$ is outside $K$. This ensures that in each ball $B_{r_K}$, the leaves of $M_N$ in the ball have boundary contained in $\partial B_{r_K}$. Then we use the uniform curvature bound in Theorem 1(c), apply Proposition B.1 in [CM5] to each ball and pass to successive subsequences on each ball, to obtain a subsequence that we rename $\{M_N\}$ that converges to a lamination, $\mathcal{L}$, with minimal leaves on compact subsets of $W \setminus \{(0, 0, t)|\, -1/2 \leq t \leq 0\}$ in the $C^\alpha$ topology for any $\alpha < 1$.

To prove Lemma 4(b) we consider the subsequence $\{M_N\}$ obtained above in Lemma 4(a) and we recall that by Theorem 1(b), $M_N \setminus \{(0, 0, t)|\, |t| \leq 1/2\} = M_{1,N} \cup M_{2,N}$ for multi-valued graphs $M_{1,N}, M_{2,N}$ over the $\{x_3 = 0\} \setminus \{0\}$ punctured plane. We cover compact subsets, $K$, of $W \setminus \{(0, 0, t)|\, |t| \leq 1/2\}$ with balls $B_{r_K}$ in the same way as in the proof of Lemma 4(a) above, such that the covering does not intersect $\{(0, 0, t)|\, |t| \leq 1/2\}$. This ensures that for all $N$, the leaves in the intersection with each ball $B_{r_K}$ are graphical over a subdomain of the $\{x_3 = 0\} \setminus \{0\}$ punctured plane. Then since by Corollary 1 (i), for all $N$, $M_{1,N} \cup M_{2,N}$ has bounded maximum distance from the $\{x_3 = 0\} \setminus \{0\}$ punctured plane , we apply Corollary 16.7 in [GT] to each ball $B_{r_K}$ to obtain uniform bounds (that are functions of $r_K$ only) on the derivatives of all orders of the graphs of the leaves in $(M_N \setminus \{(0, 0, t)|\, |t| \leq 1/2\}) \cap B_{r_K}$. Then, using standard compactness results and a diagonal argument whereby we pass to successive subsequences on each ball, we obtain a subsequence $\{M_{N_i}\}$ that converges uniformly in $C^k$ for all $k$ on compact subsets of $W \setminus \{(0, 0, t)|\, |t| \leq 1/2\}$.

□

Now, to prove Theorem 2, it is sufficient (by scaling) for us to show that there exists a sequence of compact embedded minimal disks $0 \in \Sigma_N \subset B_R \subset \mathbf{R}^3$ with $\partial \Sigma_N \subset \partial B_R$ for some $R > 0$. Theorem 1 gives us a sequence of minimal embeddings $M_N = F_N(\Omega_N) \subset \mathbf{R}^3$ with $F_N(t, 0) = (0, 0, t)$ for $|t| \leq 1/2$.

We set $R = \min\{r_0/2, 1/4\}$ where $r_0$ is given by Theorem 1(d) and we let $\Sigma_{N_i} = B_R \cap M_{N_i}$, where the sequence $N_i$ is determined by Lemma 4. We rename this sequence $\{\Sigma_N\}$.

From Theorem 1(d), we see that $\partial \Sigma_N \subset \partial B_R$. And from the properties satisfied by $M_N$ in Theorem 1, Theorem 2 (a),(b) and (c) follow immediately.

Now, we note that Corollary 1 (iii) and the smooth convergence of the leaves in Lemma 4(b), give us that the limit minimal lamination $\mathcal{L}$ in the upper hemisphere of $B_R$ consists of a leaf in the upper hemisphere, $\Sigma$, such that $\Sigma \setminus \{x_3 - \text{axis}\} = \Sigma' \cup \Sigma''$, where $\Sigma'$ and $\Sigma''$ are multi-valued graphs.

By (2.5) and (3.5), the horizontal slices $\{x_3 = x\} \cap \Sigma'$ and $\{x_3 = x\} \cap \Sigma''$ are graphs in the $\{x_3 = x\}$ plane over the line in the direction

$$(3.24) \qquad \lim_{N \to \infty} \partial_y F_N(x, 0) = \lim_{N \to \infty} (\sin u_N(x, 0), -\cos u_N(x, 0), 0).$$

We note that from (3.3), by the Cauchy-Riemann equations, $\forall N > 0$, $\partial_x u_N(x, 0) = \partial_y v_N(x, 0)$

$$(3.25) \qquad = \frac{1}{2}\left[\frac{1}{(x^2 + (\frac{1}{N})^2)^2} + \frac{1}{N}\sum_{k=1}^{N}\frac{1}{((x + k/N)^2 + (\frac{1}{N})^2)^2}\right] > 0$$

$\implies u_N(x, 0)$ is monotonically increasing w.r.t. $x$, for each fixed $N$.

Therefore, for $0 < t < R$ the angle turned by the line in (3.24) for a change in $x$ from $t$ to $2t$ is:

$$(3.26) \qquad \lim_{N \to \infty} |u_N(2t, 0) - u_N(t, 0)| = \lim_{N \to \infty}\left|\int_t^{2t} \partial_x u_N(x, 0)\, dx\right|$$

$$= \lim_{N \to \infty} \int_t^{2t} \frac{1}{2}\left[\frac{1}{(x^2 + (\frac{1}{N})^2)^2} + \frac{1}{N}\sum_{k=1}^{N}\frac{1}{((x + k/N)^2 + (\frac{1}{N})^2)^2}\right] dx$$

$$\geq \lim_{N \to \infty} \int_t^{2t} \frac{1}{2}\frac{1}{(x^2 + (\frac{1}{N})^2)^2}\, dx = \lim_{N \to \infty} \frac{N^2}{4}\left[\frac{x}{x^2 + (\frac{1}{N})^2} + N\arctan(Nx)\right]_t^{2t} =$$

$\frac{7}{48t^3}$ .

Hence we see that, for $0 < t < R$, $\{t < |x_3| < 2t\} \cap \Sigma'$ and $\{t < |x_3| < 2t\} \cap \Sigma''$ both contain an embedded $S_t$-valued graph where $S_t \geq \frac{7}{96\pi t^3} \to \infty$ as $t \to 0$ . It follows that $\Sigma'$ and $\Sigma''$ must both spiral into the $\{x_3 = 0\}$ plane. In addition, a Harnack inequality in Proposition II.2.12 in [CM2] gives a lower bound on the vertical separation of the sheets in both $\Sigma'$ and $\Sigma''$, for each
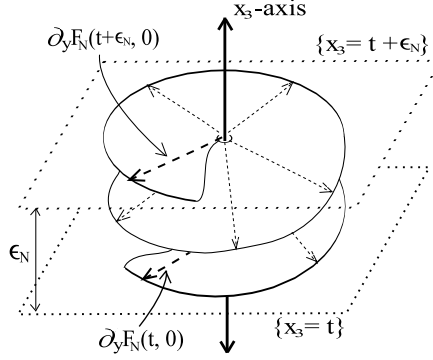
FIGURE 7. Diagram of $\Gamma_{j,N}(t)$

compact subset above the $\{x_3 = 0\}$ plane. This shows that the spiralling into the $\{x_3 = 0\}$ plane occurs with multiplicity one.

Finally, we show that the minimal lamination $\mathcal{L}$ in the lower hemisphere of $B_R \setminus \{(0,0,t)| -R \leq t \leq 0\}$ consists of a foliation by parallel planes.

We consider the sequence of embedded minimal disks $M_N$ given by Theorem 1 and we recall that by Corollary 1 (iii), $M_N \setminus \{(0,0,t)| |t| \leq 1/2\} = M_{1,N} \cup M_{2,N}$ for multi-valued graphs $M_{1,N}$, $M_{2,N}$ over the $\{x_3 = 0\} \setminus \{0\}$ punctured plane.

For arbitrary $-\frac{1}{2} \leq t \leq 0$, fixed $j = 1$ or $2$ and for all $N$ we define $\Gamma_{j,N}(t)$ to be the component of $M_{j,N}$ that is contained between the planes $\{x_3 = t\}$ and $\{x_3 = t + \epsilon_N\}$ where $\epsilon_N$ is such that the tangent vector $\partial_y F_N(t,0)$ to $M_N \cap \{x_3 = t\}$ at the $x_3$ axis (recall that this intersection is a graph in the $\{x_3 = t\}$ plane over the line in the direction $\partial_y F_N(t,0) = (\sin u_N(t,0), -\cos u_N(t,0), 0)$ by (2.5) and (3.5)) turns through an angle of $4\pi$ to the direction of the tangent vector $\partial_y F_N(t + \epsilon_N, 0)$ to $M_N \cap \{x_3 = t + \epsilon_N\}$ at the $x_3$ axis, as $x$ increases from $t$ to $t + \epsilon_N$ (See Figure 7). This definition ensures that $\Gamma_{j,N}(t) = \{t \leq x_3 \leq t + \epsilon_N\} \cap M_{j,N}$ is a graph over the $\{x_3 = 0\} \setminus \{0\}$ punctured plane such that the level sets $M_{j,N} \cap \{x_3 = x\}$ sweep out an angle of magnitude between $3\pi$ and $5\pi$ for $t \leq x \leq t + \epsilon_N$.

Now we show that $\epsilon_N \to 0$ as $N \to \infty$.

For small $s$, the angle turned by the tangent vector $\partial_y F_N(x,0)$ at the $x_3$ axis for a change in $x$ from $t$ to $t + s$ is, by (3.6):

$$|u_N(t+s,0) - u_N(t,0)| = \left| \int_t^{t+s} \partial_x u_N(x,0)\, dx \right|$$

$$= \int_{t}^{t+s} \frac{1}{2} \left[ \frac{1}{(x^2 + (\frac{1}{N})^2)^2} + \frac{1}{N} \sum_{k=1}^{N} \frac{1}{((x + k/N)^2 + (\frac{1}{N})^2)^2} \right] dx$$

$$\geq \frac{1}{2} \int_{t}^{t+s} \frac{1}{N} \sum_{k=1}^{N} \frac{1}{((x + k/N)^2 + (\frac{1}{N})^2)^2} dx$$

$$= \frac{1}{2} \frac{1}{N} \sum_{k=1}^{N} \frac{1}{((x' + k/N)^2 + (\frac{1}{N})^2)^2} ((t + s) - (t))$$ (for some $t \leq x' \leq t + s$ by the Mean Value Theorem)

$$\geq \frac{1}{2} \frac{1}{N} \frac{s}{((x' + c/N)^2 + (\frac{1}{N})^2)^2}$$ (for $1 \leq c \leq N$ chosen so that $x' + c/N \leq 1/N$)

$$\geq \frac{1}{2} \frac{1}{N} \frac{s}{((1/N)^2 + (1/N)^2)^2} = \frac{1}{8} N^3 s$$

Therefore, for any $s$, $\{t \leq x_3 \leq t + s\} \cap M_{j,N}$ contains an embedded $R_t$-valued graph where $R_t \geq \frac{1}{16\pi} N^3 s \to \infty$ as $N \to \infty$. This means that since for all N, $\Gamma_{j,N}(t) = \{t \leq x_3 \leq t + \epsilon_N\} \cap M_{j,N}$ as defined above is at most 3-valued, $\epsilon_N \leq 48\pi/N^3 \to 0$ as $N \to \infty$.

Now we have that for each $N$, $\Gamma_{j,N}(t)$ is an embedded minimal graph over the $\{x_3 = 0\} \setminus \{0\}$ punctured plane by 1(b), the boundary of each horizontal slice of $\Gamma_{j,N}(t)$ tends to infinity by Theorem 1(d), and as we have shown above, $\Gamma_{j,N}(t) = \{t \leq x_3 \leq t + \epsilon_N\} \cap M_{j,N}$ is such that $\epsilon_N \to 0$ as $N \to \infty$. Therefore, by Lemma 4(b), a subsequence $\{\Gamma_{j,N_i}(t)\}$ converges uniformly on compact subsets in the $C^k$ topology for all $k$ to an entire minimal graph minus the point $(0, 0, t)$. By a standard Bernstein type theorem, this limit graph must be a plane with a removable singularity at $(0, 0, t)$. Since $-1/2 \leq t \leq 0$ was arbitrary we have that the limit lamination $\mathcal{L}$ below the $\{x_3 = 0\}$ plane is a foliation by planes parallel to $\{x_3 = 0\}$ with removable singularities along the negative $x_3$-axis. And by intersecting with $B_R$, we obtain the required result that the lamination of the lower hemisphere of $B_R \setminus \{(0, 0, t) | -R \leq t \leq 0\}$ consists of a foliation by parallel planes, each with a removable singularity at the $x_3$-axis. The one exception is that the singularity at the origin is not removable because of the spiralling of the leaf in the upper hemisphere, $\Sigma$, into the $\{x_3 = 0\}$ plane.

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